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ASYMPTOTIC PROPERTIES OF MINIMAX SOLUTIONS OF ISAACS–BELLMAN EQUATIONS IN DIFFERENTIAL GAMES WITH FAST AND SLOW MOTIONS†

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Sufficient conditions under which the solutions of the Cauchy problem for singularly-perturbed Hamilton–Jacobi equations will converge to a limit are established. The results are used to investigate the asymptotic behaviour of the value function of a differential game involving fast and slow motions. © 1997 Elsevier Science Ltd. All rights reserved.

The Isaacs–Bellman (IB) equations for controllable systems and differential games whose dynamics contain fast and slow motions (see, for example, [1–7]) are singularly perturbed, in the sense that the Hamiltonians contain terms with coefficients of the form $1/\varepsilon$, where ε is a small parameter. It is well known that the value function of a differential game is identical with the minimax (and/or viscosity) solution of the Cauchy problem for the IB equation [8–11]. In this paper we will investigate the asymptotic behaviour of minimax solutions as $\varepsilon \to 0$. We will first establish sufficient conditions for the solutions of singularly-perturbed Hamilton–Jacobi (HJ) equations to have a limit and show that this limit is a minimax solution of the unperturbed equation. Then, using this result, we will investigate the asymptotic behaviour of value functions for similarly perturbed differential games. Use will be made of the elements of convex and non-smooth analysis [12, 13].

1. We present some information from the theory of minimax solutions of first-order partial differential equations [9, 11]. We will consider a Cauchy problem for the HJ equation

$$\partial u/\partial t + H(t, x, D_x u) = 0, \quad (t, x) \in G = (0, \theta) \times R^n \tag{1.1}$$

$$u(\theta, x) = \sigma(x), \qquad x \in R^n \tag{1.2}$$

We will assume that the function $\sigma(x)$ is continuous and that the Hamilton $H(t, x, p)$ is continuous in its domain of definition $[0, \theta] \times R^n \times R^n$; moreover, it satisfies the estimate

$$\sup_{(t,x)\in[0,\theta]\times R^n} \frac{|H(t,x,0)|}{(1+\|x\|)} < \infty \tag{1.3}$$

and the following Lipschitz conditions in p and x

$$|H(t,x,p') - H(t,x,p'')| \leqslant \lambda(x)\|p' - p''\| \tag{1.4}$$

for any $(t, x) \in [0, \theta] \times R^n, p', p'' \in R^n$ where $\lambda(x) := (1 + \|x\|)\mu$, μ being a constant

$$\sup_{(t,x,y,p)} \left\{ \frac{|H(t,x,p) - H(t,y,p)|}{\|x - y\|(1+\|p\|)} \right\} < \infty \tag{1.5}$$

for $(t, x, y, p) \in [0, \theta] \times B \times B \times R^n$, where B is an arbitrary bounded domain $B \subset R^n$.

A continuous function $\overline{G} \in (t, x) \mapsto u(t, x) \in R$ which is continuously differentiable in G and satisfies Eq. (1.2) and Eq. (1.1) for all $(t, x) \in G$ is called a classical solution of problem (1.1), (1.2). Here $\overline{G} = [0, \theta] \times R^n$; $D_x u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ is the gradient of the function u.

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As is well known, Cauchy problems and boundary-value problems for the HJ equations that occur in applications do not usually have classical solutions. One therefore introduces generalized solutions. In this paper we will use the concept of a minimax solution, which may be defined in a variety of ways; among other things, one can utilize the tools of non-smooth analysis: directional derivatives, cones of tangent directions, and sub- and super-differentials (the formulations of the definitions and proof that they are equivalent may be found, for example, in [9]).

Before presenting one of these definitions, we observe that any definition of a minimax solution contains, explicitly or implicitly, the property that the graph of such a solution is invariant with respect to what are known as characteristic inclusions, which may be introduced as follows.

Let S be some non-empty set and M a many-valued mapping

$$[0,\theta]\times R^n\times S\ni(t,x,s)\mapsto M(t,x,s)\subset R^n\times R \tag{1.6}$$

The pair (S, M) will be called a characteristic complex (or, briefly, a complex) if the following conditions hold.

1. For any $(t,x)\in[0,\theta]\times R^n$ and $s\in S$, the set $M(t,x,s)\subset R^n\times R$ is non-empty, convex and closed. For any $(t,x,s)\in[0,\theta]\times R^n\times S$ and $(f,g)\in M(t,x,s)$

$$\|f\|\leqslant\lambda(x),\qquad |g|\leqslant m(t,x)(1+\|x\|)$$

where $\lambda(x)$ is the quantity defined in (1.4). For any $s\in S$, the function $\mapsto m(t,s)$ is summable over $[0,\theta]$ and the many-valued mapping $(t,x)\mapsto M(t,x,s)$ is upper semicontinuous.

2a. For any $(t,x)\in[0,\theta]\times R^n$ and $p\in R^n$

$$\max_{s\in S}\min\{\langle f,p\rangle-g:(f,g)\in M(t,x,s)\}=H(t,x,p)$$

2b

$$\max_{s\in S}\min\{\langle f,p\rangle-g:(f,g)\in M(t,x,s)\}=H(t,x,p)$$

Denote the set of complexes (S, M) by $C(H)$. We note that the conditions are satisfied, for example, by the pair (S, M) where $S = R^n$ and

$$M(t,x,s)=\{(f,g)\in R^n\times R:\|f\|\leqslant\lambda(x),\quad g=\langle f,s\rangle-H(t,x,s)\}$$
$$(t,x)\in\overline{G},\quad s\in R^n$$

where $\lambda(x) = (1+\|x\|)\mu$ is the same as in the Lipschitz condition (1.4).

Choose a complex $(S, M)\in C(H)$ and $s\in S$ arbitrarily. The symbol $\mathrm{Sol}(t_0, x_0, z_0, s)$ will denote the set of absolutely continuous functions $x(\cdot), z(\cdot)): [0,\theta]\mapsto R^n\times R$. that satisfy the condition $(x(t_0), z(t_0)) = (x_0, z_0)$ and the differential inclusion

$$(\dot{x}(t),\dot{z}(t))\in M(t,x(t),s) \tag{1.7}$$

Definition 1. A minimax solution of Eq. (1.1) is a continuous function $[0,\theta]\times R^n\ni(t,x)\mapsto u(t,x)\in R$ that satisfies the following condition of weak invariance with respect to (1.7): for any $(t_0,x_0,z_0)\in\mathrm{gr}\,u$, $s\in S$ and $\tau\in[t_0,\theta]$ a trajectory $(x(\cdot), z(\cdot))\in\mathrm{Sol}(t_0,x_0,z_0,s)$ exists such that $(\tau, x(\tau), z(\tau))\in\mathrm{gr}\,u$.

We will call (1.7) a characteristic inclusion. It is known that this definition is independent of the choice of a complex $(S, M)\in C(H)$.

It can also be shown that minimax solutions and viscosity solutions in the sense of Crandall and Lions [10] are equivalent. Under our assumptions concerning the Hamiltonian and the terminal function σ, one and only one minimax solution of the Cauchy problem (1.1), (1.2) exists. The proofs of these facts may be found in [9, 11].

The concepts of upper and lower solutions play a major role in the theory of minimax solutions. We present the definitions in a form convenient for further use in this paper.

As before, S will be some non-empty set and M a many-valued mapping of type (1.6). Call the pair

(S, M) an upper (lower) characteristic complex if conditions 1 and 2a (1 and 2b) are satisfied. The set of upper (lower) characteristic complexes will be denoted by $C^{\uparrow}(H)$ ($C^{\downarrow}(H)$).

Definition 2. An upper (lower) solution of Eqs (1.1) is a lower (upper) semicontinuous function $[0, \theta] \times R^n \ni (t, x) \mapsto u(t, x) \in R$ satisfying the following condition: For any $(t_0, x_0, z_0) \in \operatorname{epi} u((t_0, x_0, z_0) \in \operatorname{hypo} u)$, $s \in S$ and $\tau \in [t_0, \theta]$, a trajectory $(x(\cdot), z(\cdot)) \in \operatorname{Sol}(t_0, x_0, z_0, s)$ exists such that $(\tau(x(\tau)), z(\tau)) \in \operatorname{epi} u((\tau, x(\tau), z(\tau)) \in \operatorname{hypo} u)$. Here $(S, M) \in C^{\uparrow}(H)$ ($(S, M) \in C^{\downarrow}(H)$), $\operatorname{Sol}(t_0, x_0, z_0, s)$ is the set of trajectories of the differential inclusion (1.7) that satisfy the condition $(x(t_0), z(t_0)) = (x_0, z_0)$.

The symbols epi u and hypo u denote the sets

$$\{(t,x,z) : z \geqslant u(t,x), (t,x) \in \overline{G}\}, \quad \{(t,x,z) : z \leqslant u(t,x), (t,x) \in \overline{G}\}$$

respectively—the epigraph and hypograph of the function u. The definition of an upper (lower) solution is independent of the choice of the complex $(S, M) \in C^{\uparrow}(H)$ $(S, M) \in C^{\downarrow}(H)$. It is known that a function u is a minimax solution if and only if it is simultaneously an upper and a lower solution.

2. We will consider two Cauchy problems, unperturbed and perturbed

$$\partial u / \partial t + H(t, x, D_x u) = 0, \quad u(\theta, x) = \sigma(x) \tag{2.1}$$

$$\partial u_\varepsilon / \partial t + H_\varepsilon(t, x, y, D_x u_\varepsilon, D_y u_\varepsilon) = 0, \quad u_\varepsilon(\theta, x, y) = \sigma(x) \tag{2.2}$$

In these equations, as before, $t \in (0, \theta), x \in R^n, y \in R^1$ is a new variable, ε is a positive real parameter, and the function u_ε depends on variables $(t, \tilde{x})$, where $\tilde{x} = (x, y)$. Accordingly, the Hamiltonian H_ε depends on the variables $(t, \tilde{x}, \tilde{p})$, where $\tilde{p} = (p, r)$, $t \in R^1$. It is assumed that the function $\sigma(x)$ is continuous, and the Hamiltonians $H(t, x, p)$ and $H_\varepsilon(t, \tilde{x}, \tilde{p})$ satisfy the conditions indicated in Section 1.

Let us examine the conditions under which the solutions u_ε of problem (2.2) converge as $\varepsilon \to 0$ to a solution u of problem (2.1). To that end, we introduce further constructions.

Let $\rho > 0$, $(S, M) \in C^{\uparrow}(H)$ or $(S, M) \in C^{\downarrow}(H)$. The symbol $G^\rho(t_0, \tau, x_0, z_0, s)$ will denote the attainability set of the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in M(t, x(t), s) + B_\rho. \tag{2.3}$$

where $s \in S$, $0 \leqslant t_0 < \tau \leqslant \theta$, (x_0, z_0) is the initial position; $B_\rho = \{(f, g) \in R^n \times R : \|f\|^2 + g^2 \leqslant \rho^2\}$. Note that $(x^*, z^*) \in G^\rho(t_0, \tau, x_0, z_0, s)$ if and only if a trajectory of the differential inclusion (2.3) exists such that $(x(t_0), z(t_0)) = (x_0, z_0)$, $(x(\tau), z(\tau)) = (x^*, z^*)$. Speaking somewhat freely, we will also refer to the attainability set of the complex (S, M^ρ).

Let $(S_\varepsilon, M_\varepsilon) \in C^{\downarrow}(H_\varepsilon)$ or $(S_\varepsilon, M_\varepsilon) \in C^{\uparrow}(H_\varepsilon)$, $s' \in S_\varepsilon$. The symbol $G_\varepsilon(t_0, \tau, x_0, y_0, z_0, s')$ will denote the attainability set of the differential inclusion

$$(\dot{x}(t), \dot{y}(t), \dot{z}(t)) \in M_\varepsilon(t, x(t), y(t), s') \tag{2.4}$$

Note that $G_\varepsilon(\ldots)$ is a set in $R^n \times R^l \times R$. We will also say that $G_\varepsilon(\ldots)$ is the attainability set of the complex $(S_\varepsilon, M_\varepsilon)$.

Condition 1 (*Condition* 2). A complex $(S, M) \in C^{\uparrow}(H)$ (a complex $(S, M) \in C^{\downarrow}(H)$) and a compact set $Y \subset R^1$ exists such that

$$G^\rho(t_0, \tau, x_0, z_0, s) \times Y \supseteq \bigcup_{y_0 \in Y} G_\varepsilon(t_0, \tau, x_0, y_0, z_0, \psi_\varepsilon(s)) \tag{2.5}$$

where $G_\varepsilon(\ldots)$ is the attainability set of a certain complex $(S_\varepsilon, M_\varepsilon) \in C^{\uparrow}(H_\varepsilon)$ ($(S_\varepsilon, M_\varepsilon) \in C^{\downarrow}(H_\varepsilon)$). It is assumed here that for any number $\varepsilon > 0$ and point $(t_*, x_*) \in (-\infty, \theta) \times R^n$ we have a well-defined mapping $s \mapsto \psi_\varepsilon(s)$: $S \mapsto S_\varepsilon$ and quantities $\rho = \rho(\varepsilon, t_*, x_*) > 0$, $\delta = (\varepsilon, t_*, x_*) > 0$ such that $\lim \rho(\varepsilon, t_*, x_*) = 0$, $\lim \delta(\varepsilon, t_*, x_*) = 0$ as $\varepsilon \to 0$. Condition (2.5) must hold for all $\varepsilon > 0$, $s \in S$, $z_0 \in R$, $(t_0, x_0) \in B(t_*, x_*; \alpha)$ $(\alpha = \alpha(t_*, x_*) \in (0, \theta - t_*))$, $\tau \in [t_0 + \delta(\varepsilon, t_*, x_*), \theta]$. The symbol $B(t_*, x_*; \alpha)$ will denote the closed sphere in $R \times R^n$ of radius α with centre at the point (t_*, x_*).

This formal condition will serve as a sufficient condition for solutions of singularly perturbed equations to approach an asymptotic limit. We will then consider examples of differential games in which this condition is fairly easy to verify. As will be evident from the examples, the compact set Y in (2.5) contains the domain of attraction of the fast variables.

Theorem 1. Let Conditions 1 and 2 be satisfied. In addition, let the set Y in Condition 1 be the same as the Y in Condition 2. Let $u_\varepsilon(\varepsilon > 0)$ be minimax solutions of problem (2.2). Then for any $(t, x, y) \in [0, \theta] \times R^n \times Y$ a limit

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon(t, x, y) = u(t, x) \tag{2.6}$$

exists, and the function u: $[0, \theta] \times R^n \mapsto R$ defined by this limit relationship is a minimax solution of problem (2.1).

We first prove a few auxiliary propositions.

Proposition 1. Suppose Condition 1 is satisfied. Define

$$v_\varepsilon^0(t, x) = \min_{y \in Y} v_\varepsilon(t, x, y) \tag{2.7}$$

where $v_\varepsilon(t, x, y)$ is an upper solution of problem (2.2). Let $B(t_*, x_*; \alpha)$ be a closed sphere in $(0, \theta) \times R^n$, which is defined by Condition 1. Then, for any $(t_0, x_0) \in B(t_*, x_*; \alpha)$, $z_0 \geqslant v_\varepsilon^0(t_0, x_0)$, $s \in S$, $\tau \in (t_0 + \delta, \theta]$, a point $(x^*, z^*) \in G^p(t_0, \tau, x_0, z_0, s)$ exists such that $(\tau, x^*, z^*) \in \operatorname{epi} v_\varepsilon^0$.

Proof. Choose $y_0 \in Y$ so that

$$v_\varepsilon(t_0, x_0, y_0) = \min_{y \in Y} v_\varepsilon(t_0, x_0, y) = v_\varepsilon^0(t_0, x_0)$$

By assumption, $z_0 \geqslant v_\varepsilon^0(t_0, x_0)$, and so $(t_0, x_0, y_0, z_0) \in \operatorname{epi} v_\varepsilon$. Since v_ε is an upper solution of problem (2.2), a point

$$(x^*, y^*, z^*) \in G_\varepsilon(t_0, \tau, x_0, y_0, z_0, \psi_\varepsilon(s))$$

exists such that $z^* \geqslant v_\varepsilon(\tau, x^*, y^*)$. It follows from (2.5) that

$$(x^*, y^*, z^*) \in G^p(t_0, \tau, x_0, y_0, z_0) \times Y$$

Since $y^* \in Y$, it follows that $v_\varepsilon(\tau, x^*, y^*) \geqslant v_\varepsilon^0(\tau, x^*)$. Hence we obtain $z^* \geqslant v_\varepsilon^0(\tau, x^*)$.

Proposition 2. The function

$$v^h(t, x) = \liminf_{\substack{\varepsilon \downarrow 0 \\ (t', x') \to (t, x)}} v_\varepsilon^0(t', x') \tag{2.8}$$

is an upper solution of problem (2.1).

Proof. It can be shown that the function v^h takes finite values. This function is lower semicontinuous and satisfies the condition $v^h(\theta, x) = \sigma(x)$ (see the proofs of the analogous statements in [9, 11]).

Let $(S, M) \in C^\uparrow(H)$ be the complex occurring in Condition 1. We choose $(t_0, x_0) \in [0, \theta) \times R^n$, $z_0 \geqslant v^h(t_0, x_0)$, $\tau \in (t_0, \theta]$, s, $\in S$ arbitrarily. We will show that $(x(\cdot), z(\cdot)) \in \operatorname{Sol}(t_0, x_0, z_0, s)$ exists such that $(\tau, x(\tau), z(\tau)) \in \operatorname{epi} v$ or, what is the same

$$\{\tau\} \times G(t_0, \tau, x_0, z_0, s) \cap \operatorname{epi} v^h \neq ... \tag{2.9}$$

where $G(\ldots)$ is the attainability domain of the complex (S, M).

Let $(t_0, x_0) \in \operatorname{int} B(t_*, x_*; \alpha)$ (int B is the interior of the sphere B). Let us first consider the case $z_0 > v^h(t_0, x_0)$. In that case, by (2.8), a sequence $(\tau_k, t_k, x_k)_{k=1}^\infty$ exists, such that $z_0 \geqslant v_{\varepsilon_k}^0(t_k, x_k)$, $\lim_{k\to\infty} \varepsilon_k = 0$, $\lim_{k\to\infty}(t_k, x_k) = (t_0, x_0)$. We may assume that $(t_k, x_k) \in B(t_*, x_*; \alpha)$ and $t_k + \delta(\varepsilon_k, t_*, x_*)) < \tau$ for all $k = 1, 2, \ldots$. Using Proposition 2, we see a sequence $(x^k, z^k) \in G^{\rho^k}(t_k, \tau, x_k, z_0, s)$ exists such that $(\tau, x^k, z^k) \in \operatorname{epi} v^0_{\varepsilon_k}$. Here, by condition 1, $\rho^k = \rho(\varepsilon_k) \to 0$ as $k \to \infty$. We may assume without loss of generality that the limit $\lim_{k\to\infty}(x^k, z^k) = (x^*, z^*)$ exists. Note that

$$(x^*, z^*) \in G(t_0, \tau, x_0, z_0, s).$$

Since $z^k \geq v^h_{\varepsilon_k}(\tau, x^k)$, it follows by the definition of the function v^h that $z^* \geq v^h(\tau, x^*)$. Thus, we have proved (2.9) in the case $z_0 > v^h(t_0, x_0)$.

In the case $z_0 \geq v^h(t_0, x_0)$, we consider the sequence $z_k = z_0 + 1/k$. Since $z_k > v^h(t_0, x_0)$, it follows from what has been proved that

$$\{\tau\} \times G(t_0, \tau, x_0, z_k, s) \cap \operatorname{epi} v^h \neq ...$$

Passing to the limit, using the fact that the set epi v^h is closed and that the mapping $z \mapsto G(t_0, \tau, x_0, z, s)$ is upper semicontinuous, we again get (2.9).

Suppose that Condition 2 is satisfied. Let $W_\varepsilon(\varepsilon > 0)$ be lower solutions of problem (2.2). Consider the functions

$$w^0_\varepsilon(t,x) = \max_{y\in Y} w_\varepsilon(t,x,y), \quad w^h(t,x) = \limsup_{\substack{\varepsilon\downarrow 0 \\ (t',x')\to(t,x)}} w^0_\varepsilon(t',x') \tag{2.10}$$

It can be shown that w^h is a lower solution of problem (2.1). The proof is analogous to that of Propositions 1 and 2.

Let $u_\varepsilon(t, x, y)$ $(\varepsilon > 0)$ be minimax solutions of problem (2.2). Put $v_\varepsilon = w_\varepsilon = u_\varepsilon$ in (2.7), (2.8) and (2.10). Define the corresponding functions v^h and w^h. By construction, $w^h \geq v^h$.

On the other hand, as v^h is an upper solution and w^h a lower solution of problem (2.1), it follows [9–11] that $w^h \leq v^h$. Therefore $v^h = w^h = u$, where u is a minimax solution of problem (2.1) (recall that a minimax solution can be defined as a function that is simultaneously an upper and a lower solution).

We thus obtain

$$u(t,x) = \limsup_{\substack{\varepsilon\downarrow 0 \\ (t',x')\to(t,x)}} \max_{y\in Y} u_\varepsilon(t',x',y) = \liminf_{\substack{\varepsilon\downarrow 0 \\ (t',x')\to(t,x)}} \min_{y\in Y} u_\varepsilon(t',x',y) \tag{2.11}$$

This implies (2.6) and completes the proof of Theorem 1.

3. *Example* 1. Consider the following unperturbed differential game and its Hamiltonian

$$\begin{aligned} &\dot{x} = f(t, x, p, q), \quad p \in P, q \in Q \\ &\gamma = \sigma(x(\vartheta)) - \int_{t_0}^{\vartheta} g(t, x(t), p(t), q(t))dt \\ &H(t,x,\zeta) = \min_{p\in P}\max_{q\in Q}[\langle f(t,x,p,q),\zeta\rangle - g(t,x,p,q)] = \\ &= \max_{q\in Q}\min_{p\in P}[\langle f(t,x,p,q),\zeta\rangle - g(t,x,p,q)] \end{aligned} \tag{3.1}$$

Consider the singularity perturbed differential game

$$\begin{aligned} &\dot{x} = f(t,x,y_1,y_2) \\ &\dot{y}_1 = (1/\varepsilon)[p' - y_1] = h_{\varepsilon 1}(y_1, p'), \quad \dot{y}_2 = (1/\varepsilon)[q' - y_2] = h_{\varepsilon 2}(y_2, q'), \quad p' \in P, \ q' \in Q \\ &\gamma = \sigma(x(\vartheta)) - \int_{t_0}^{\vartheta} g(t, x(t), y_1(t), y_2(t))dt \end{aligned} \tag{3.2}$$

The Hamiltonian in this game is defined by

$$\begin{aligned} &H_\varepsilon(t,x,y_1,y_2,\zeta,\zeta_1,\zeta_2) = \langle f(t,x,y_1,y_2),\zeta\rangle - g(t,x,y_1,y_2) + \\ &+(1/\varepsilon)\left[\min_{p\in P}\langle\zeta_1,p\rangle - \langle\zeta_1,y_1\rangle\right] + (1/\varepsilon)\left[\max_{q\in Q}\langle\zeta_2,q\rangle - \langle\zeta_2,y_2\rangle\right] \end{aligned} \tag{3.3}$$

where P and Q are convex compact sets and the functions $f(t,x,p,q), g(t,x,p,q)$ are continuous and satisfy a Lipschitz condition with respect to (x,p,q). The function $\sigma(x)$ is continuous, $t \in [0,\theta], x \in R^n$.

In this example Condition 1 is satisfied. The upper characteristic complexes may be chosen here as follows:

$$S = Q,\ s = q,\ S_\varepsilon = Q,\ s' = q,\ \psi_\varepsilon(q) = q$$

$$M(t,x,q) = \mathrm{co}\{(f(t,x,p,q),\ g(t,x,p,q)) : p \in P\}$$

$$M_\varepsilon(t,x,y_1,y_2,q') = \mathrm{co}\{(f(t,x,y_1,y_2),\ h_{\varepsilon 1}(y_1,p'),\ h_{\varepsilon 2}(y_2,q'),\ g(t,x,y_1,y_2)) : p' \in P\}$$

To choose lower characteristic complexes, one simply interchanges the controls p and q. The compact set Y in Conditions 1 and 2 may be defined as $P \times Q = Y$.

Under the fact that P and Q are strongly invariant with respect to the subsystems of fast variables $y_1^\varepsilon(t)$ and $y_2^\varepsilon(t)$, respectively, and also using estimates similar to the Gronwall inequality [14] for the norm of the difference between the slow variable $x^\varepsilon(t)$ and the solution of the differential inclusion

$$\dot{x}(t) = f(t,x(t),y_1^\varepsilon(t),q) \in \mathrm{co}\, f(t,x(t),P,q),\quad x(t_0) = x^\varepsilon(t_0) = x_0$$

we obtain the following values of the parameters in Condition 1

$$\delta(\varepsilon) = \varepsilon^\eta,\quad \eta < 1$$

$$\rho(\varepsilon) = L\,\mathrm{diam}\, Q e^{L\theta}[(e^{L\delta(\varepsilon)} - 1) + e^{-\delta(\varepsilon)/\varepsilon}] \to 0 \quad \text{as} \quad \varepsilon \to 0$$

where L is the Lipschitz constant for the functions $f(t,x,p,q), g(t,x,p,q)$ with respect to x, p, q.

Analogous estimates, with q and p, Q and P interchanged, are obtained for the verification of Condition 2.

4. The sufficient Conditions 1 and 2 may be modified so as to include new classes of problems (2.1), (2.2). We introduce the following definition.

A compact set $D \in [0,\theta] \times R^n$ is said to be *strongly invariant and compatible with a Hamiltonian H* if D is strongly invariant with respect to the differential inclusion $\|\dot{x}\| \leqslant \lambda(x)$, where $\lambda(x)$ is the quantity in condition (1.4), defined for the Hamiltonian H.

We modify Conditions 1 and 2 as follows.

Condition 3 (*Condition* 4). A complex $(S,M) \in C^\uparrow(H)$ (a complex $(S,M) \in C^\downarrow(H)$) exists such that, for any compact set $D \subset [0,\theta] \times R^n$ which is strongly invariant and compatible with the Hamiltonian H, a compact set $Y(D) \subset R^l$ exists such that

$$G^\rho(t_0,\tau,x_0,z_0,s) \times Y(D) \supseteq \bigcup_{y_0 \in Y(D)} G_\varepsilon(t_0,\tau,x_0,y_0,z_0,\psi_\varepsilon(s)) \tag{4.1}$$

where $G_\varepsilon(\ldots)$ is the attainability set of some complex $(S_\varepsilon, M_\varepsilon) \in C^\uparrow(H_\varepsilon)$ ($(S_\varepsilon, M_\varepsilon) \in C^\downarrow(H_\varepsilon)$). It is assumed here that for any $\varepsilon > 0$ there are a well-defined mapping $s \mapsto \psi_\varepsilon(s): S \mapsto S_\varepsilon$ and quantities $\rho = \rho(\varepsilon) > 0$, $\delta = \delta(\varepsilon) > 0$ such that $\lim \rho(\varepsilon) = 0$, $\lim \delta(\varepsilon) = 0$ as $\varepsilon \to 0$. Relation (2.5) may hold for all $\varepsilon > 0$, $s \in S$, $z_0 \in R$, $(t_0,x_0) \in D$, $\tau \in [t_0 + \delta(\varepsilon), \theta]$.

Using this condition and following the same scheme as in Theorem 1, one can prove that the minimax solutions $v_\varepsilon(t,x,y)$ of the perturbed problem (2.2) converge to a minimax solution $u(t,x)$ of the unperturbed problem (2.1) for any $(t,x,y) \in D \times Y(D)$.

Example 2. Consider the singularly perturbed differential game

$$\dot{x} = f(t,x,y),\quad \varepsilon\dot{y} = -y + \xi(t,x,\alpha,\beta),\quad \alpha \in A, \beta \in B \tag{4.2}$$

$$\gamma = \sigma(x(\vartheta)) - \int_{t_0}^{\vartheta} g(t,x(t),y(t))dt \tag{4.3}$$

The Hamiltonian in this game is defined by

$$H_\varepsilon(t,x,y,\zeta,\zeta_1) = \langle f(t,x,y),\zeta\rangle - g(t,x,y) - -(1/\varepsilon)\langle y,\zeta_1\rangle + (1/\varepsilon)\psi(t,x,\zeta_1) \tag{4.4}$$

$$\psi(t,x,\zeta_1) = \min_{\alpha\in A}\max_{\beta\in B}\langle\zeta_1,\xi(t,x,\alpha,\beta)\rangle = \max_{\beta\in B}\min_{\alpha\in A}\langle\zeta_1,\xi(t,x,\alpha,\beta)\rangle \tag{4.5}$$

The unperturbed differential game and its Hamiltonian are

$$\dot{x} = f(t,x,\xi(t,x,\alpha,\beta)), \quad \alpha \in A, \ \beta \in B$$
$$\gamma = \sigma(x(\vartheta)) - \int_{t_0}^{\vartheta} g(t,x(t),\xi(t,x,\alpha,\beta))dt \tag{4.6}$$

where

$$Y(t,x,\alpha) = \mathrm{co}\{\xi(t,x,\alpha,\beta)): \beta \in B\}$$
$$Y(t,x,\beta) = \mathrm{co}\{\xi(t,x,\alpha,\beta)): \alpha \in A\} \tag{4.7}$$

It is assumed that the functions $f(\cdot)$, $g(\cdot)$, $\xi(\cdot)$, $\sigma(\cdot)$, $\partial\xi(\cdot)/\partial t$, $\partial\xi(\cdot)/\partial x$ are continuous, L_f is the Lipschitz constant for $f(\cdot)$ and $g(\cdot)$ with respect to x, y, and L is the Lipschitz constant for $\xi(\cdot)$ with respect to x.

In this problem Conditions 3 and 4 are satisfied. The upper characteristic complexes may be chosen as follows:

$$S = B, \ s = \beta, \ s_\varepsilon = B, \ s' = \beta, \ \psi_\varepsilon(\beta) = \beta$$
$$M(t, x, \beta) = \mathrm{co}\{(f(t, x, \xi), \ g(t, x, \xi)) : \xi \in Y(t, x, \beta)\}$$
$$M_\varepsilon(t, x, y, \beta) = \mathrm{co}\{(f(t, x, y), (1/\varepsilon)(-y + \xi), g(t, x, y)) : \xi \in Y(t, x, \beta)\}$$

For the choice of the lower characteristic complexes, the controls α and β are interchanged.

The compact set $Y(D)$ is defined as follows:

$$Y(D) = \mathrm{co}\{\xi(t, x, \alpha, \beta) : (t, x) \in D, \ \alpha \in A, \beta \in B\} + B_1$$
$$B_1 = \{y \in R^l : \|y\| \leqslant 1\}$$

Let $r_\beta^\varepsilon[t]$ be the distance between the fast variable $y_\varepsilon(t)$ and the set $Y(t, x^\varepsilon(t), \beta)$. Using an estimate for the rate of change of the function $r_\beta^\varepsilon[t]$ along a motion of system (4.2), and also using estimates similar to the Gronwall inequality [1] for the norm of the distance between the slow variable $x^\varepsilon(t)$ and the solution $x(t)$ of the differential inclusion

$$\dot{x}(t) \in \mathrm{co}\{f(t,x(t),\xi): \xi \in Y(t,x(t),\beta)\}, \quad x(t_0) = x^\varepsilon(t_0) = x_0$$

we get the following values of the parameters in Condition 3

$$\delta(\varepsilon) = \varepsilon^\eta, \quad \eta < 1$$
$$\rho(\varepsilon) = L_f \ \ \mathrm{diam}\, U(D) \ e^{L_f(1+L_\xi)\theta}[(e^{L_f(1+L_\xi)\delta(\varepsilon)} - 1) + e^{-\delta(\varepsilon)/\varepsilon}] \to 0$$

as $\varepsilon \to 0$.

Analogous values are obtained in the verification of Condition 4.

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